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Minimal H-sets and Unicity of Best Approximations

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INTRODUCTION

Let X be a compact Hausdorff space and let $C(X)$ denote the Banach space of real-valued functions continuous on X, normed with the sup norm. Let V be a linear subspace of $C(X)$. For $p \in V$ and $f \in C(X)$, let $E(p, f)$ denote the set of extreme points of p-f, $E(p, f) = \{x \in X \mid |p(x) - f(x)| = \|p - f\|\}$. Following Collatz [2], we say a set $Q \subseteq X$ is an H-set for V if $Q = P \cup N$, $P \cap N = \varnothing$ and there is no $q \in V$ satisfying $q(x) > 0$, $x \in P$ and $q(x) < 0$, $x \in N$.

We can decompose $E(p, f)$ into two disjoint sets P and N by $P = \{x \in E(p, f) \mid p(x) - f(x) > 0\}$, $N = \{x \in E(p, f) \mid p(x) - f(x) < 0\}$, and we call this the natural decomposition. The following theorem is well known and characterizes best approximations to f by elements of V.

THEOREM. *p is best approximation to f if and only if $E(p, f)$ is an H-set for V under the natural decomposition.*

It is our purpose to give a sufficient condition, in terms of $E(p, f)$, that a best approximation be unique.

MINIMAL H-SETS

Let $Q = P \cup N$ be an H-set for V. We call Q a minimal H-set if for each $x' \in Q$, $Q' = Q - \{x'\}$ is not an H-set, where $Q' = P' \cup N'$ and $P' \subseteq P$, $N' \subseteq N$. Minimal H-sets correspond to primitive extremal signatures introduced by Rivlin and Shapiro [4]. They have been enumerated and characterized by Brosowski [1] and Taylor [5] for the case when X is a compact subset of R^n and V is the $(n + 1)$-dimensional subspace of linear polynomials.

LEMMA 1. *Let p be a best approximation to f, and let $Q \subseteq E(p,f)$ be a minimal H-set under the natural decomposition. Then all best approximations agree on Q.*

Proof. Let $Q = P \cup N$ and suppose q is also a best approximation to f. Suppose $q(x') \neq p(x')$ where $x' \in Q$. With no loss of generality, we may assume $x' \in P$, and, thus, $q(x') < p(x')$. As Q is a minimal H-set, there is $h \in V$ such that $h(x) > 0$ for $x \in P - \{x'\}$ and $h(x) < 0$ for $x \in N$. Choose $\lambda > 0$ so that $0 \leqslant \lambda \mid h(x')| < p(x') - q(x')$. Since $p(x) - q(x) \geqslant 0$ for $x \in P - \{x'\}$ and $p(x) - q(x) \leqslant 0$ for $x \in N$, then $p(x) - q(x) + \lambda h(x)$ is positive on P and negative on N, a contradiction. Thus, $q(x) = p(x)$ for all $x \in Q$.

In [3], Newman and Shapiro defined the concept of a set of uniqueness. Following the terminology of that paper, we say a minimal H-set Q is a strong minimal H-set if $q \in V$ and $q(x) = 0$ for all $x \in Q$ implies $q \equiv 0$. Using Lemma 1 and the characterization of best approximations given in the introduction, the following is immediate.

COROLLARY 1. *If $E(p,f)$ contains a strong minimal H-set under the natural decomposition then p is a best approximation to f and p is unique.*

When V is finite dimensional, it is possible to characterize strong minimal H-sets. Let $\mid Q \mid$ denote, as usual, the cardinality of the set Q. Then we have the following

THEOREM 1. *Suppose V has dimension n and suppose Q is a minimal H-set for V. Q is a strong minimal H-set if and only if $\mid Q \mid = n + 1$.*

Proof. Let $Q = P \cup N$ and let $\{u_1(x),..., u_n(x)\}$ be a basis for V. Define $\sigma(x) = 1$ for $x \in P$ and $\sigma(x) = -1$ for $x \in N$, and let $U(x) = (u_1(x),..., u_n(x))$. As Q is an H-set, the origin θ of R^n is in the convex hull of the set of vectors $A = \{\sigma(x) \cdot U(x) \mid x \in Q\}$. By Carathéodory's theorem, θ is in the convex hull of $n + 1$ or fewer vectors in A, so if Q is a minimal H-set it must be that $\mid Q \mid \leqslant n + 1$.

Let $\mid Q \mid = k$ and form the $k \times n$ matrix $M = (m_{ij})$, where $m_{ij} = u_j(x_i)$. The set A of vectors is linearly dependent, so when $k \leqslant n$, there is a nonzero vector $y \in R^n$ such that $My = \theta$ (where θ is the origin in R^k). Thus, for $k \leqslant n$, Q is not a strong minimal H-set.

Suppose $k = n + 1$. If the vectors in A span R^{j-1} but not R^j where $j \leqslant n$, we can again apply Carathéodory's theorem to deduce Q is not minimal. Thus, M contains a nonsingular $n \times n$ submatrix, and, therefore, Q is a strong minimal H-set.

References

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